MATH 5C - FINAL EXAM v1 Summer 2023

200 POINTS

NAME: ________________

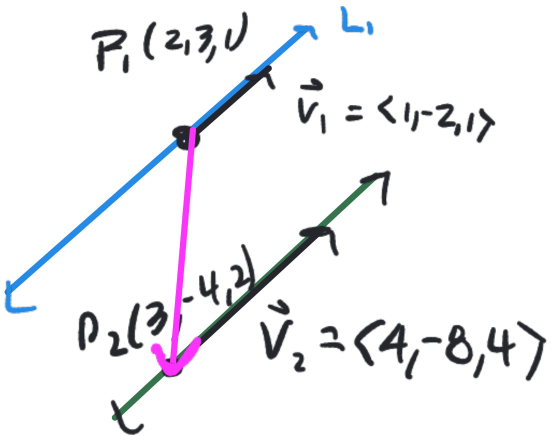
Instructions on Canvas. SHOW **ALL** WORK. Each problem worth 20 points.

(1)

(a) Find the equation of the plane containing the lines.

$$L_1\begin{cases} x=2+t\\ y=3-2t\\ z=1+t\end{cases} \qquad L_2\begin{cases} x=3+4s\\ y=-4-8s\\ z=2+4s\end{cases}$$

For plane, need point and normal vector



Notice $\vec{v}_1 = \frac{1}{2}\vec{V}_2$ so $v_1 \parallel V_2$.

Form new vector $\vec{P_1P_2} = \langle 1,-7,1\rangle$

Then $\vec{n} = \vec{P_1P_2} \times \vec{V_1} = \langle -5, 0, 5\rangle$

(Remember - it's easy to check cross product since $\vec{n}$ should be orthog. to both $\vec{P_1P_2}$ and $\vec{v}$)

Point $(2,3,1)$

Plane: $-5(x-2)+0(y-3)+5(z-1)=0$

$-5x+5z+5=0$

$x-z-1=0$

(b) Find an equation for the tangent plane to the surface $x=y^2+z^2+1$ at the point $(3,1,-1)$

Express surface as $\underbrace{x-y^2-z^2-1}_{F(x,y,z)}=0$

Then $\vec{\nabla}F(3,1,-1)$ is orthogonal to the level surface of F at $(3,1,-1)$

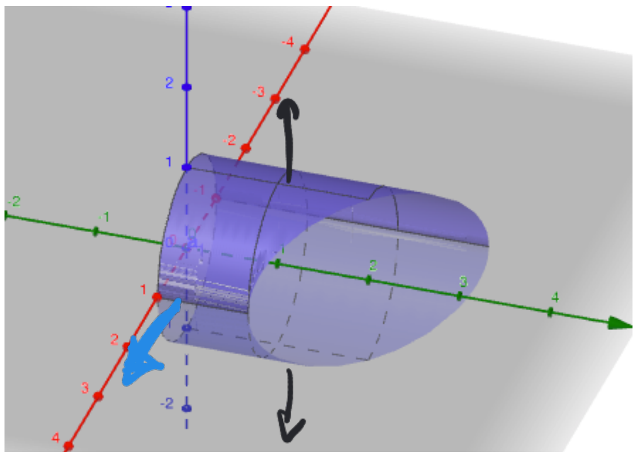
$\vec{\nabla}F = \langle 1,-2y,-2z\rangle$

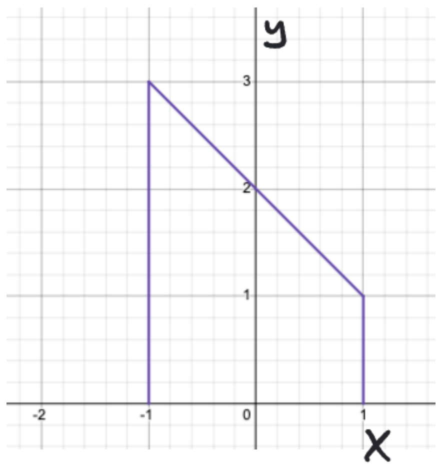
$\vec{n} = \vec{\nabla}F(3,1,-1) = \langle 1,-2,2\rangle$

$x-3-2(y-1)+2(z+1)=0$

$x-2y+2z+1=0$

(2) Given: $\vec{F}(x,y,z)=\langle x,\ z,\ 0\rangle$, and surface S which is portion of the cylinder $x^2+z^2=1$, bounded by the planes $y=0$ and the plane $x+y=2$ oriented outward,as shown. Note: this surface in not closed. It is the cylindrical sides only. Evaluate the flux, $\iint_S \vec{F}\bullet d\vec{S}$





Using Parametric Surfaces

$\vec{F}(\cos\theta\ ,\ y, \sin\theta)$ $\quad 0\le\theta\le 2\pi$

$0\le y\le 2-x=2-\cos\theta$

$\vec{r}_\theta=\langle -\sin\theta, 0, \cos\theta\rangle$

$\vec{r}_y=\langle\ 0\ ,\ 1\ ,\ 0\ \rangle$

$\vec{r}_\theta\times\vec{r}_y=\langle -\cos\theta, 0, -\sin\theta\rangle$

* need outward

(note for $\theta=0$, x comp. should be + ...see graph)

$-\vec{r}_\theta\times\vec{r}_y=\langle\cos\theta, 0, \sin\theta\rangle$

$\vec{F}=\langle\cos\theta, \sin\theta, 0\rangle$

$\vec{F}\bullet\vec{r}_\theta\times\vec{r}_y=\cos^2\theta$

$$\iint_S \vec{F}\bullet d\vec{S}=\int_0^{2\pi}\int_0^{2-\cos\theta}\cos^2\theta\,dy\,d\theta$$

$$=\int_0^{2\pi}\cos^2\theta(2-\cos\theta)\,d\theta$$

$$=\int_0^{2\pi}2\cos^2\theta-\cos^3\theta\,d\theta$$

$$=2\int_0^{2\pi}\cos^2\theta\,d\theta-\int_0^{2\pi}(1-\sin^2\theta)\cos\theta\,d\theta$$

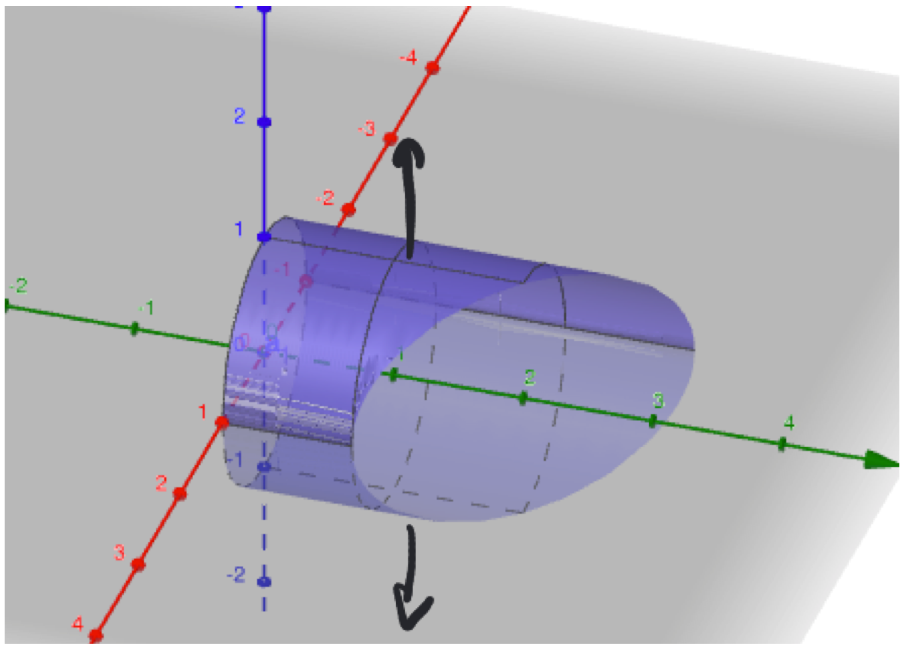
$u=\sin\theta$

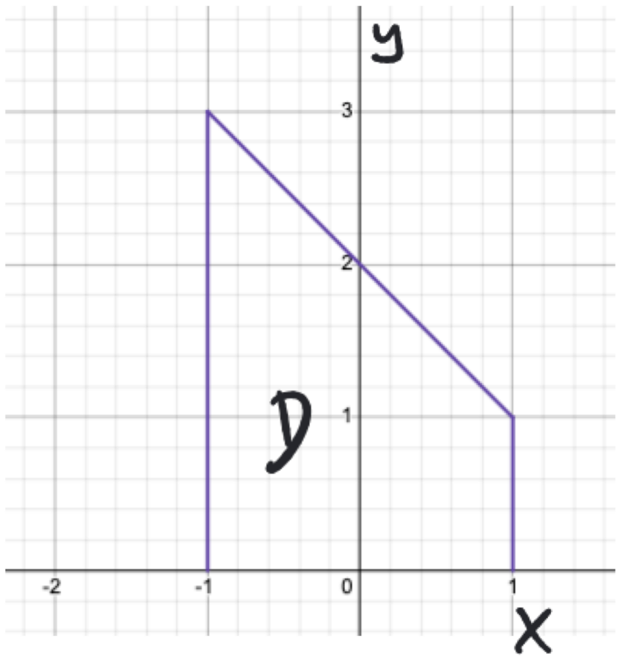
$$=2\left(\tfrac{1}{2}\theta+\tfrac{1}{4}\sin 2\theta\right)\Big]_0^{2\pi}-\sin\theta+\frac{\sin^3\theta}{3}\Big]_0^{2\pi}$$

$$=\boxed{2\pi}$$

over for another solution

(2) Given: $\vec{F}(x,y,z)=\langle x,\ z,\ 0\rangle$, and surface S which is portion of the cylinder $x^2+z^2=1$, bounded by the planes $y=0$ and the plane $x+y=2$ oriented outward, as shown. Note: this surface in not closed. It is the cylindrical sides only. Evaluate the flux, $\iint_S \vec{F}\bullet d\vec{S}$





Without Parametric surfaces, do top and bottom separately

Top (upward)

$$z=\sqrt{1-x^2}$$

$$\underbrace{z-\sqrt{1-x^2}}_{G(x,y,z)}=0$$

$$\vec{\nabla}G=\left\langle \frac{x}{\sqrt{1-x^2}},\ 0,\ 1\right\rangle$$

$$\vec{F}=\left\langle x,\ \sqrt{1-x^2},\ 0\right\rangle$$

$$\vec{F}\bullet\vec{\nabla}G=\frac{x^2}{\sqrt{1-x^2}}$$

Bottom (downward)

$$z=-\sqrt{1-x^2}$$

$$\underbrace{z+\sqrt{1-x^2}}_{G}=0$$

$$\vec{\nabla}G=\left\langle \frac{-x}{\sqrt{1-x^2}},0,1\right\rangle$$

$$-\vec{\nabla}G=\left\langle \frac{x}{\sqrt{1-x^2}},0,-1\right\rangle$$

$$\vec{F}=\left\langle x,-\sqrt{1-x^2},0\right\rangle$$

$$\vec{F}\bullet-\vec{\nabla}G=\frac{x^2}{\sqrt{1-x^2}}$$

$$\iint_S \vec{F}\cdot d\vec{S}=\iint_{Top}\vec{F}\cdot d\vec{S}+\iint_{Botton}\vec{F}\cdot d\vec{S}$$

$$=\iint_D \vec{F}\cdot\vec{\nabla}G\,dA+\iint_D \vec{F}\cdot-\vec{\nabla}G\,dA$$

$$=\iint_D \frac{x^2}{\sqrt{1-x^2}}+\frac{x^2}{\sqrt{1-x^2}}\,dA=\int_{-1}^{1}\int_0^{2-x}\frac{2x^2}{\sqrt{1-x^2}}\,dy\,dx$$

cont'd

$$= \int_{-1}^{1}\int_{0}^{2-x} \frac{2x^2}{\sqrt{1-x^2}}\,dy\,dx$$

$$= \int_{-1}^{1} (2-x)\frac{2x^2}{\sqrt{1-x^2}}\,dx$$

$$= \int_{-1}^{1} \frac{4x^2}{\sqrt{1-x^2}}\,dx - \underbrace{\int_{-1}^{1} \frac{2x^3}{\sqrt{1-x^2}}\,dx}_{\text{This is zeo}}$$

$\int_{-a}^{a} \text{odd fcn} = 0$

$$= \int_{-1}^{1} \frac{4x^2}{\sqrt{1-x^2}}\,dx$$

Trig. substitution

$x = \sin\theta \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = \cos\theta\,d\theta$

$$= \int_{-\pi/2}^{\pi/2} \frac{4\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} 4\sin^2\theta\,d\theta = 4\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big]_{-\frac{\pi}{2}}^{\pi/2}$$

$$= 4\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \boxed{2\pi}$$

(3) The position vector of a particle is $\vec{r}(t) = \langle t^2, \ln t, 2t \rangle$ $\quad \vec{r}\,' = \langle 2t, \frac{1}{t}, 2 \rangle$

(a) Find the length of the curve for $1 \le t \le 2$

(b) Find parametric equations of the line tangent to $\vec{r}(t)$ at t=1.

a) Length = $\int_C 1\, ds$ (line integral of scaler function)

$$ds = \|\vec{r}\,'(t)\|\, dt$$
$$= \sqrt{4t^2 + \frac{1}{t^2} + 4}\, dt$$
$$= \sqrt{\left(2t + \frac{1}{t}\right)^2}\, dt$$
$$= 2t + \frac{1}{t}$$

$$\int_1^2 \left(2t + \frac{1}{t}\right) dt = t^2 + \ln|t| \Big]_1^2 = 4 + \ln 2 - (1 + \ln 1)$$
$$= \boxed{3 + \ln 2}$$

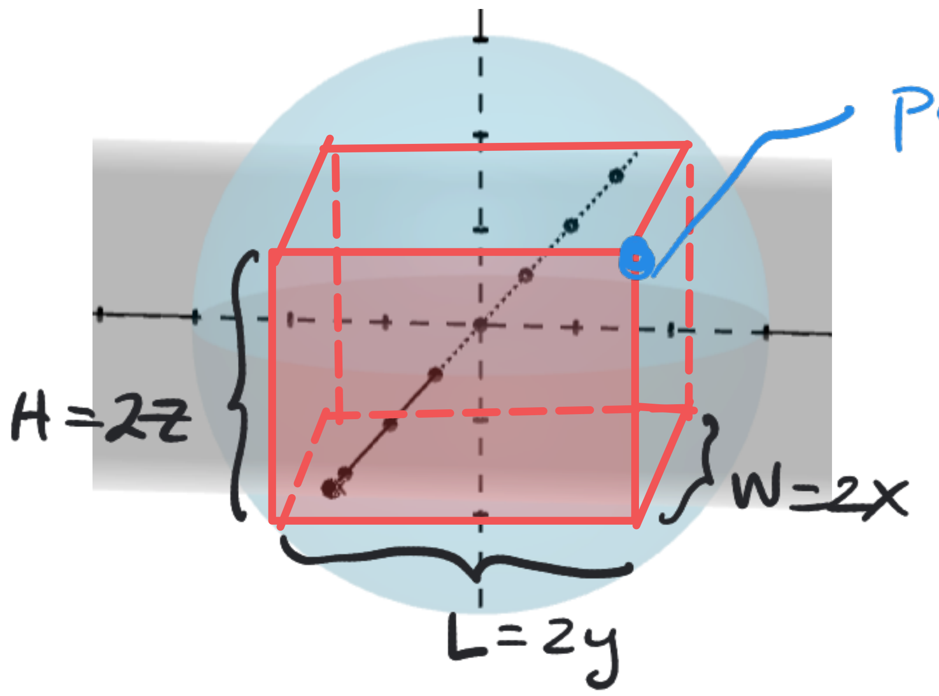
b) For line, need point an direction vector

point: $\vec{r}(1) = \langle 1, 0, 2 \rangle$

$\vec{v} = \vec{r}\,'(1) = \langle 2, 1, 2 \rangle$

$$\boxed{\begin{cases} x = 1 + 2t \\ y = t \\ z = 2 + 2t \end{cases}}$$

(4) Find the maximum volume of a rectangular box that can be inscribed in a sphere of radius 3. Show how you know it is an absolute maximum.



point (x,y,z) on sphere with $x,y,z>0$

Maximize $V = LWH$

$= 2y\,2x\,2z$

$= 8xyz$

Subject to constraint $x^2+y^2+z^2=9$

You can use methods from 14.7 where you solve the constraint for a variable $\left(z=\sqrt{9-x^2-y^2}\right)$ and put it into $V=8xyz$

OR

<u>Lagrange Multipliers</u>

$\vec{\nabla} V = \lambda \vec{\nabla} g$
$g(x,y,z)=0$
$\Rightarrow \begin{cases} 8yz = \lambda 2x \\ 8xz = \lambda 2y \\ 8xy = \lambda 2z \\ x^2+y^2+z^2=9 \end{cases}$

Need to be organized and keep system together

$\Rightarrow \begin{cases} 8xyz = \lambda 2x^2 \\ 8xyz = \lambda 2y^2 \\ 8xyz = \lambda 2z^2 \\ x^2+y^2+z^2=9 \end{cases}$ these are all equal

Note: if $\lambda = 0$, then $x, y,$ or z must be zero, but $x,y,z>0$ so $\lambda \neq 0$

$$\Rightarrow \quad \begin{cases} \lambda 2x^2 = \lambda 2y^2 = \lambda 2z^2 \\ x^2+y^2+z^2=9 \end{cases}$$

$\lambda \neq 0$ so can divide out

$$\begin{cases} x^2 = y^2 = z^2 \\ x^2+y^2+z^2=9 \end{cases} \Rightarrow \quad x^2+x^2+x^2 = 9$$

$$3x^2 = 9$$

$$x^2 = 3$$

$$x = \pm\sqrt{3}$$

$$y = \pm\sqrt{3}$$

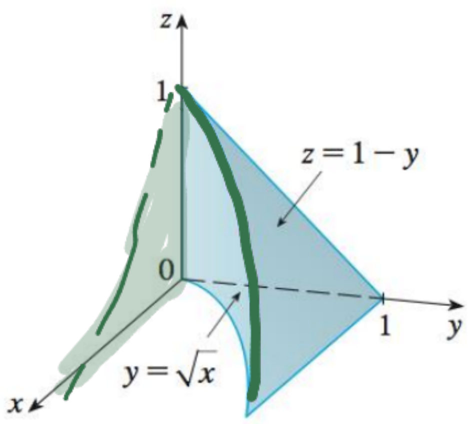
$$z = \pm\sqrt{3}$$

$x, y, z > 0 \Rightarrow \quad x = y = z = \sqrt{3}$

$$\max V = 8xyz = \boxed{24\sqrt{3}}$$

Since physically, we know there must be a box of maximum volume which can be inscribed in a sphere, and since this is the only point to be considered, it must yield the abs max which is $24\sqrt{3}$

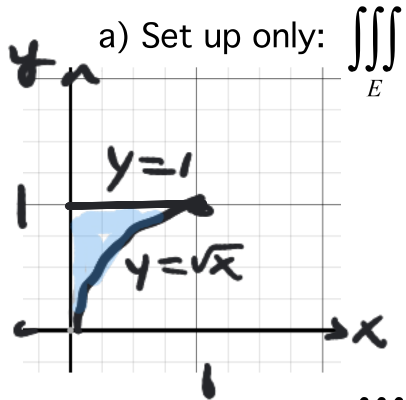
(5) Let E be the solid shown.



Intersection

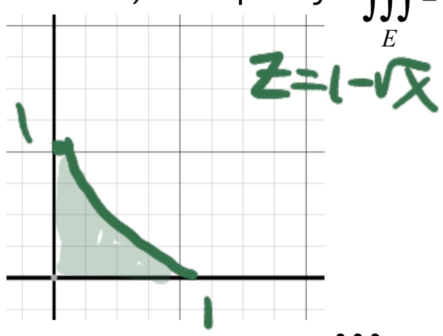
$$\begin{cases} z = 1-y \\ y = \sqrt{x} \end{cases} \xrightarrow{\text{elim } y} \quad z = 1-\sqrt{x}$$

a) Set up only: $\iiint_E z\,dV$ Triple integral- rectangular coordinates; order dz dy dx



$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} z \; dz\,dy\,dx$$

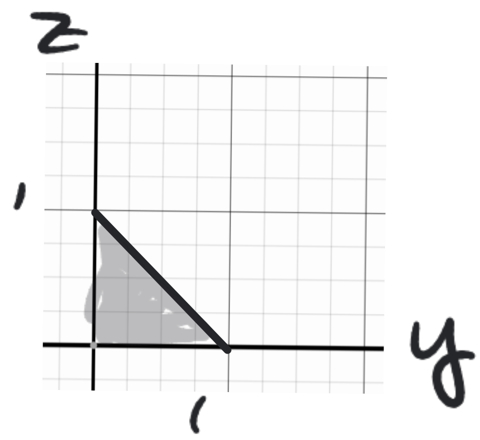
b) Set up only: $\iiint_E z\,dV$ Triple integral- rectangular coordinates; order dy dz dx



$$\int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-y} z \; dy\,dz\,dx$$

c) Compute $\iiint_E z\,dV$ using any order you wish. (Note, some orders are messier than others).

Many options:



x first has a simple projection, and the back surface is x = 0. Also, no x in integrand to start which keeps things simpler

$$\int_0^1 \int_0^{1-y} \int_0^{y^2} z \; dx\,dz\,dy$$

$1 - 2y + y^2$

$$= \int_0^1 \int_0^{1-y} z y^2 \, dz\,dy$$

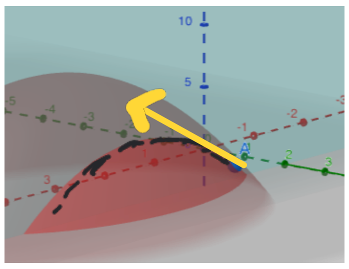
$$= \int_0^1 \frac{1}{2} y^2 (1-y)^2 \, dy$$

$$= \frac{1}{2}\int_0^1 (y^2 - 2y^3 + y^4)\,dy = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \boxed{\frac{1}{60}}$$

(6) A hiker at the point (1,2,1) on the hill $z = 6x - x^2 - y^2$ (where the z axis points up, the y axis north, the x axis east)

$\frac{\partial z}{\partial x} = 6 - 2x \quad \frac{\partial z}{\partial y} = -2y$

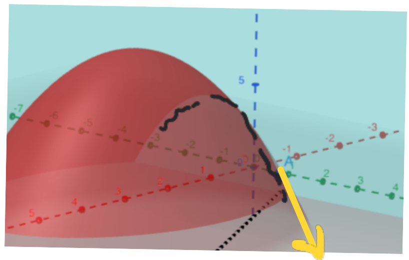
(a) Find $\left.\frac{\partial z}{\partial x}\right|_{(1,2)}$. Explain what this represents physically.



$\left.\frac{\partial z}{\partial x}\right|_{(1,2)} = \boxed{4}$

Moving from the point (1,2,1) eastward, you are going uphill at a rate of 4 vertical feet per 1 horizontal ft.

(b) If the hiker heads north from the point (1,2,1), will she be going up the hill or down? at what rate?



Y axis is north, so

$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \boxed{-4}$

Down hill.

(c) If the hiker heads in the direction from (1,2) towards (5,5) is she going up the hill or down? at what rate?

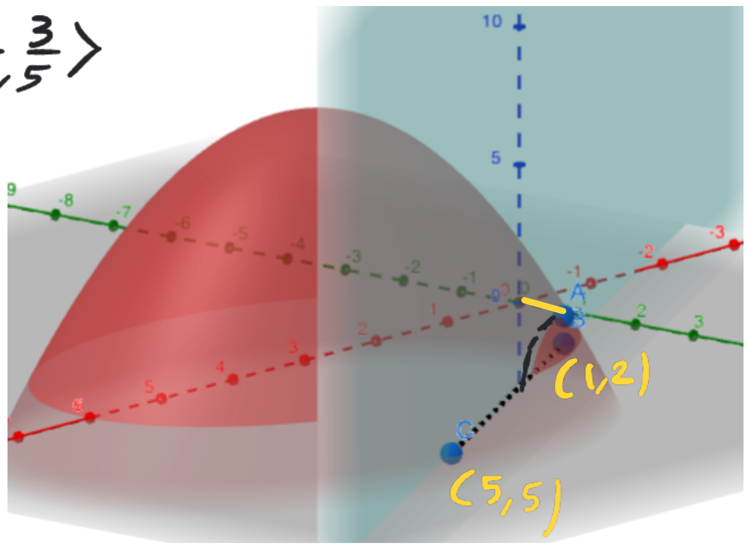
P Q

$\vec{v} = \vec{PQ} = \langle 4,3 \rangle \quad \vec{u} = \langle \frac{4}{5}, \frac{3}{5} \rangle$

$\vec{\nabla} f = \langle 6-2x, -2y \rangle$

$\vec{\nabla} f(1,2) = \langle 4, -4 \rangle$

$D_{\vec{u}} f(1,2) = \vec{\nabla} f(1,2) \cdot \vec{u} = \frac{16}{5} - \frac{12}{5} = \boxed{\frac{4}{5}}$



(d) What is the direction of steepest climb?

In the direction of the gradient

$\boxed{\langle 4, -4 \rangle}$

(7) Given $f(x,y) = e^x - xe^y$

a) Find all local extrema and saddle points.

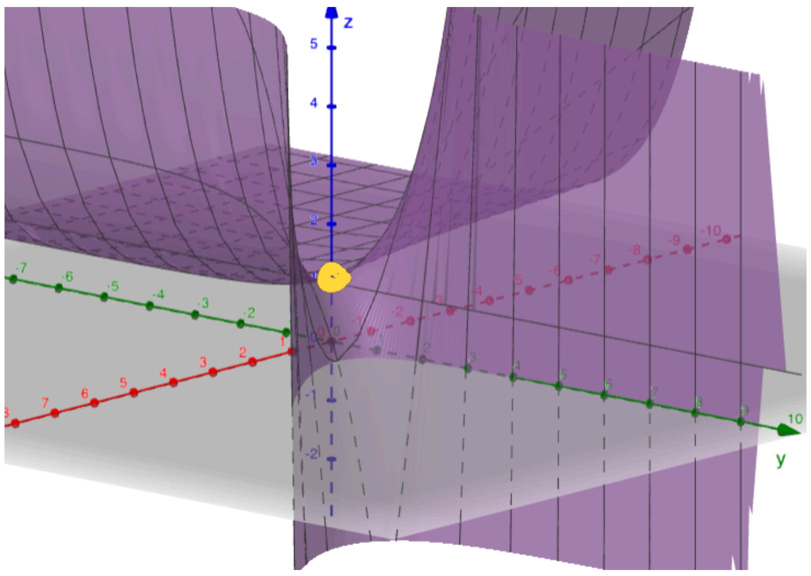
Critical points $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$ $\begin{cases} e^x - e^y = 0 \\ -xe^y = 0 \end{cases} \Rightarrow x = 0$ since $e^y \neq 0$

$(0,0)$ is only crit pt. $\left\{ e^0 - e^y = 0 \Rightarrow e^y = 1 \Rightarrow y = 0 \right.$

$$D = \begin{vmatrix} e^x & -e^y \\ -e^y & -xe^y \end{vmatrix} = -xe^x e^y - e^{2y} = -e^y(e^x + e^y)$$

$D(0,0) < 0 \Rightarrow \boxed{\text{saddle at } (0,0)}$

Reasonable



b) Compute $\int_0^1 \int_0^4 f(x,y)\,dy\,dx$

$$\int_0^1 \int_0^4 (e^x - xe^y)\,dy\,dx$$

$$= \int_0^1 ye^x - xe^y \Big]_0^4 = \int_0^1 4e^x - xe^4 - (-x)\,dx$$

$$= \int_0^1 (4e^x - xe^4 + x)\,dx$$

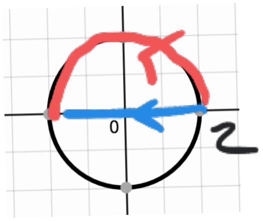
$$= 4e^x - \tfrac{1}{2}x^2 e^4 + \tfrac{1}{2}x^2 \Big]_0^1$$

$$= 4e - \tfrac{1}{2}e^4 + \tfrac{1}{2} - (4)$$

$$= \boxed{4e - \tfrac{1}{2}e^4 - \tfrac{7}{2}}$$

(8 and 9) Given the vector field $\vec{F}(x,y)=\langle 6x+y,\, x-2y\rangle$ and the curve C given by

$\vec{r}=\langle 2\cos t, 2\sin t\rangle \quad 0\le t\le \pi$ $\quad \vec{r}(0)=\langle 2,0\rangle \quad \vec{r}(\pi)=\langle -2,0\rangle$



Compute the work $\int_C \vec{F}\bullet d\vec{r}$ two different ways. Be sure to explain clearly what method you are using. (Not just two different parameterizations for the same curve)

(8) $\frac{\partial}{\partial x}(x-2y)=1 \qquad \frac{\partial}{\partial y}(6x+y)=1$

so $\vec{F}$ is conservative

① Fundamental Theorem

we can find $f(x,y)=3x^2+xy-y^2+C$

$\int_C \vec{F}\bullet d\vec{r} = f(-2,0)-f(2,0)= 0$

② Use a simpler path

Line segment : $\vec{r}=\langle 2-4t, 0\rangle \quad 0\le t\le 1$

$\vec{r}\,'=\langle -4, 0\rangle$

$\vec{F}=\langle 6(2-4t), 2-4t\rangle$

$\int_C \vec{F}\bullet d\vec{r} = \int_0^1 -48+96t\, dt = -48t+48t^2\Big]_0^1 = 0$

(9)

③ Direct

$\vec{r}\,'=\langle -2\sin t, 2\cos t\rangle \quad 0\le t\le \pi$

$\vec{F}=\langle 12\cos t+2\sin t,\, 2\cos t-4\sin t\rangle$

$\vec{F}\cdot\vec{r}\,' = -24\sin t\cos t-4\sin^2 t+4\cos^2 t-8\cos t\sin t$

$= -32\sin t\cos t+4(\cos^2 t-\sin^2 t)$

$= -32\sin t\cos t+4\cos 2t$

$\int_C \vec{F}\bullet d\vec{r} = \int_0^\pi (-32\sin t\cos t+4\cos 2t)\,dt$

$-16\sin^2 t + 2\sin 2t\Big]_0^\pi = 0$

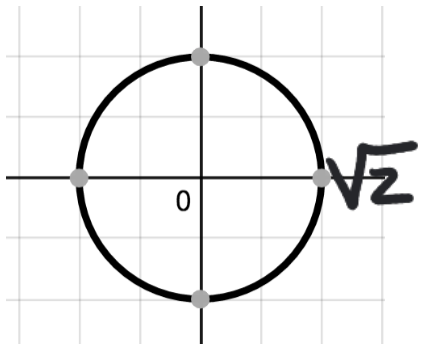
(10) Find the volume of the solid bounded by the paraboloids $z = 2x^2 + 2y^2$ and.

$z = 6 - x^2 - y^2$

Intersection $\begin{cases} z = 2x^2 + 2y^2 \\ z = 6 - x^2 - y^2 \end{cases} \Rightarrow$ $6 - x^2 - y^2 = 2x^2 + 2y^2$

$6 = 3(x^2 + y^2)$

$x^2 + y^2 = 2$



$$V = \iiint_E dV$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{2r^2}^{6-r^2} dz \, r \, dr \, d\theta$$

$$= 2\pi \int_0^{\sqrt{2}} (6r - 3r^3) \, dr$$

$$= 2\pi \left(3r^2 - \frac{3}{4} r^4\right]_0^{\sqrt{2}}$$

$$= 2\pi (6 - 3)$$

$$= \boxed{6\pi}$$